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Solutions of the Ginsparg-Wilson Relation

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Abstract

We analyze general solutions of the Ginsparg-Wilson relation for lattice Dirac operators and formulate a necessary condition for such operators to have non-zero index in the topologically nontrivial background gauge fields.

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Recently there are very interesting developments in theoretical understandings of the chiral symmetry on the lattice. The idea stems from the Ginsparg-Wilson (GW) relation [1] which was derived in 1981 as the remnant of chiral symmetry on the lattice after blocking a chirally symmetric theory with a chirality breaking local renormalization group transformation. The original GW relation is

$$D\gamma_5 + \gamma_5 D = 2aD\gamma_5 RD, \tag{1}$$

where D is lattice Dirac operator, R is a non-singular hermitian operator which is local in the position space and trivial in the Dirac space, and a is the lattice spacing which reminds us the fact that D becomes chirally symmetric in the continuum limit $a \to 0$. According to the Nielsen-Ninomiya theorem [2] the chiral symmetry of a local Dirac operator defined on the regular lattice must be broken in order to avoid the species doubling. The main advantage of the GW relation is that it introduces the chiral symmetry breaking of D in the mildest way [1]. Although it does not ensure the absence of the species doubling, it does incorporate two remarkable properties in the following.

The first is that the action $A = \bar{\psi} D \psi$ has an exact symmetry :

$$\begin{aligned} \psi &\to \exp[i\theta\gamma_5(\mathbb{1} - RD)]\psi, && (2)\\ \bar{\psi} &\to \bar{\psi}\exp[i\theta(\mathbb{1} - DR)\gamma_5], && (3) \end{aligned}$$

where θ is a global parameter, which was discovered by Lüscher [3]. The second is that any operator D satisfying the GW relation possesses a well defined integer index on a finite lattice [4], [3]

$$\lim_{\epsilon \to 0} \epsilon \sum_n \langle \overline{\psi}_n \gamma_5 \psi_n \rangle_f = \mathrm{Tr}(\gamma_5 RD) = n_- - n_+ \equiv \text{index } D, \tag{4}$$

where l.h.s. stands for the fermionic average of $\overline{\psi}\gamma_5\psi$ calculated with the infinitesimal mass ϵ added to the operator D, and n_+ (n_-) are the number of the zero modes of D with positive (negative) chiralities. This is in contrast to the Wilson-Dirac operator for which the l.h.s. generally is not an integer on a finite lattice.

It is essentially due to these two properties that such formulations of lattice QCD can possess the attractive features pointed out in [3]-[6]. However only the GW relation itself is not sufficient to guarantee that any D satisfying (1) must possess exact zero modes with definite chiralities, and reproduce the Atiyah-Singer index theorem on the lattice. In this paper, we analyze general solutions of GW relation and formulate a necessary condition for them to have non-zero indices in topologically non-trivial background gauge fields. We limit our consideration to the operators D satisfying the hermiticity property

$$D^\dagger = \gamma_5 D \gamma_5. \tag{5}$$

First, we consider the case of non-singular D which is relevant to topologically trivial gauge field background, except possibly some 'exceptional' configurations. Then eq. (1) is equivalent to the following equation linear in D^{-1},

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = 2a\gamma_5 R, \tag{6}$$

and its general solution can be written in the form

$$D = (\mathbb{1} + aD_c R)^{-1} D_c = D_c(\mathbb{1} + aRD_c)^{-1} \tag{7}$$

where D_c is the chirally symmetric lattice Dirac operator, i.e.

$$D_c\gamma_5 + \gamma_5 D_c = 0 \tag{8}$$

Thus in the nonsingular case the problem of constructing explicit solutions of D reduces to finding a proper realization of the chirally symmetric operator D_c. Note that by virtue of the condition (5) and eq. (8) the operator D_c is antihermitian, and therefore, normal. In order to avoid species doubling for D defined on a regular lattice, D_c should be non-local. Additional limitations to the form of D_c come from the requirement of the locality of D. For a more detailed discussion on the properties of D_c we refer to our paper [7] where a few explicit examples are also presented.

It is interesting to observe that in this case both D_c and D can be constructed from a unitary operator V ($V^\dagger = V^{-1}$) which satisfies the hermiticity condition

$$\gamma_5 V \gamma_5 = V^\dagger \tag{9}$$

Indeed, for any given chirally invariant D_c satisfying (5), D_c is antihermitian, so that $\det(aD_c + \mathbb{1}) \neq 0$, then there exists a unitary operator

$$V = (aD_c - \mathbb{1})(aD_c + \mathbb{1})^{-1} \tag{10}$$

satisfying (9). So D_c can be represented as

$$D_c = a^{-1}\frac{\mathbb{1} + V}{\mathbb{1} - V}, \tag{11}$$

provided that $\mathbb{1} - V$ is nonsingular. Substituting (11) into (7), we obtain the general solution of (6) for nonsingular D in terms of the unitary operator V

$$\begin{aligned} D &= a^{-1}[(\mathbb{1} + V)R + \mathbb{1} - V]^{-1}(\mathbb{1} + V) && (12) \\ &= a^{-1}(\mathbb{1} + V)[R(\mathbb{1} + V) + \mathbb{1} - V]^{-1} && (13) \end{aligned}$$

Note that in contrast to eq. (11) these expressions make sense even when operator $\mathbb{1} - V$ is singular. As we will show later, due to this fact eqs. (12) and (13) represent a class of solutions of (1) also for singular D, and thus are valid for any gauge configurations.

Let us now consider the case when D is singular, i.e. $\det D = 0$, which is the result we would like to have in the topologically non-trivial gauge field background. In this case we are interested in only those D which have the possibility to reproduce the index theorem on the lattice, i.e. have a non-zero index in (4). So we obtain a necessary condition for any solutions of (1) to possess non-zero indices in topologically non-trivial background gauge fields. We hope that this condition not only serves as a discriminant to rule out any unphysical solutions of GW relation but also can provide guidelines to construct viable solutions of GW relation. We state our result in the following theorem.

Theorem: For any lattice Dirac operator D satisfying hermiticity condition (5) and the GW relation (1), the necessary condition for it to have non-zero index in the topologically nontrivial gauge background is

$$\det(\mathbb{1} - aDR) = 0 \tag{14}$$

or, equivalently

$$\det(\mathbb{1} - aRD) = 0 \tag{15}$$

Proof : Assume that $\det(\mathbb{1} - aDR) \neq 0$. Then there exists a chirally symmetric normal operator

$$D_c = (\mathbb{1} - aDR)^{-1} D. \tag{16}$$

It is obvious that any zeromode of D is a zeromode of D_c, and vice versa. Therefore index D_c = index D. However according to the theorem proved in [8], the index of any chirally symmetric normal Dirac operator is zero, so index D_c = index $D = 0$. Since $\det(\mathbb{1} - aDR) = \det(\mathbb{1} - aRD)$, this completes the proof.

In other words, we have proved that in order the operator D to have nonzero index the chirally invariant operator D_c in (16), and therefore in (7), should not exist. However D is still well defined and exists. Eqs. (11) and (12) suggest a simple interpretation of this seemingly paradoxial situation. As shown in [9], for any unitary operator V satisfying (9), if it has real (± 1) eigenmodes then these real eigenmodes are chiral and the total chirality of all real eigenmodes must vanish

$$n_{+1}^{+} - n_{+1}^{-} + n_{-1}^{+} - n_{-1}^{-} = 0 \tag{17}$$

where n_{+1}^{+} (n_{+1}^{-}) denotes the number of positive (negative) chirality eigenmodes of eigenvalue $+1$, while n_{-1}^{+} (n_{-1}^{-}) denotes the number of positive (negative) chirality eigenmodes of eigenvalue -1. The -1 eigenmodes of V correspond to the zero modes of D. Thus, if D has non-zero index ($n_{-1}^{-} - n_{-1}^{+} \neq 0$), then $n_{+1}^{+} - n_{+1}^{-} \neq 0$ and V has eigenvalue $+1$. Then the chirally invariant operator D_c in (11) is no longer defined, while the operator D in (12) becomes singular but still well defined. Therefore (12) is indeed a class of general solutions for the GW relation (1) for any gauge configurations.

Finding a unitary operator V which can have eigenvalues $+1$ and -1 in the topologically non-trivial sectors, however, is a highly non-trivial task. So far we know only one explicit example of V which does satisfy this requirement. It is the unitary operator derived from the overlap formalism [5],

$$V = D_w (D_w^\dagger D_w)^{-1/2}, \tag{18}$$

where D_w is the standard Wilson-Dirac operator but with a negative mass in the range $(-2a^{-1}, 0)$. In [9], it has been demonstrated that this solution of GW relation indeed reproduces exact zero modes and the index theorem is satisfied exactly on a finite lattice. The zero modes are also in very good agreement with the continuum theory. At this moment we cannot provide another example of V which can satisfy all our requirements.

It is instructive to note that solutions of the GW relation may have zero index not only because zero modes with opposite chiralities always appear in pairs but also because they may not have any zero modes at all. Consider the naive massless Dirac fermion operator D_n on the regular lattice, the random lattice [10] and the random-block lattice [11] respectively. Since D_n is chirally symmetric, it can be taken as D_c and the Dirac operator D can be constructed from (7). For any one of these GW-Dirac operators, we do not find any genuine zero modes in any topologically non-trivial sectors.

To summarize, we have demonstrated that the GW relation does not guarantee the existence of exact zero modes nor the realization of index theorem on the lattice. If a solution of GW relation, D, in topologically nontrivial sector gives $\det(\mathbb{1} - aRD) \neq 0$, and therefore can be expressed in terms of a chirally invariant operator D_c, its index must be zero, and thus it should be dropped from the list of viable lattice Dirac fermion operators. We note in passing that the necessary condition (14) or (15) is precisely the condition that the generators of the lattice chiral transformations in (2) and (3) are singular. The general solution for the GW relation (1) is obtained in (12).

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